Chance and strategy in Poker

Laure Elie* Romuald Elie†

September 2007

Summary

The aim of this analysis is to quantify the impact of chance versus strategy in the game of *Texas Hold'em Poker*. It thereby complements N. Alon's [1] work on this subject by broadening the game model considered. The results were obtained with a theoretical study carried out by digital simulations of virtual poker games. We concluded that, for a sufficiently high number of consecutive games, it is clear that strategy rather than chance is the overiding factor in the outcome of a *Texas Hold'em Poker* game.

Key words: *Texas Hold'hem Poker*, chance, strategy, Monte Carlo simulations.

* Professeur, Université Paris Diderot, Laboratoire de Probabilités et Modèles aléatoires, UMR CNRS 7599, elie@math.jussieu.fr

† Maître de conférence, ENSAE & Université Paris Dauphine, elie@ensae.fr

1 Introduction

Poker being a game increasingly practised, a question with important legal consequences has come to the fore over the last few years: *Is poker a game where strategy prevails over chance?* This study endeavours to give an answer which is mathematically rigourous to the question detailed from a legal point of view in [7].

Of course, a very large number of studies suggest various poker strategies (see [3] for example), but very few deal with the question of chance in games results. They are mainly focused on game theory problematics searching for strategic balance between players, or artificial intelligence allowing a progressive adaptation to opponents' behaviour. The only study which seems to answer this question satisfactorily is N. Alon’s [1]. Studying a simplified version of a game of poker, his analysis concludes that Poker is essentially a game of strategy. Indeed, thanks to the Central Limit Theorem, a powerful tool in the theory of probability, he shows that the strategy employed is a determining factor in the result of a sufficiently large number of games.

In this study, we first validate N. Alon's findings and we then generalise them. Indeed, N. Alon considers a simplified version of the game of poker, taking into account only the last round of the game where all the community cards are known. He studies mainly games between two players: Alice who has a well-defined strategy and Bob who plays in a random manner. He assumes that Alice has an intimate knowledge of the way Bob plays which gives her a considerable advantage. We put ourselves in a real game of *Texas Hold'em poker* model, in following the pattern of the various stages of the game: Preflop, Flop, Turn, River. In this more general model, even if we only consider two players, the analytical calculations carried out by N. Alon can not be done so we have used digital simulations. In other words we have performed a digital simulation of virtual poker games between Bob, who plays in a random manner, and Alice who follows a well thought-out strategy, and we have analysed the results.

In this model which is very close to reality, we draw conclusions that are very similar to N. Alon's: **for a sufficiently high number of games, the strategy employed is a determining factor in the outcome of a game of** *Texas Hold’em Poker*.

This study is organised as follows: first, we present the rules and we describe the course of a *Texas Hold’em Poker* game. Then, we study N. Alon's results and we suggest a more general game model, for which we specify interesting strategies. Finally, with the assistance of virtual games digital simulations, we estimate the probability of winning using these strategies. The strategies selected are not necessarily the best ones in the end, but they have the advantage of defining simple and realistic decision criteria for a poker player who is able to assess his own skills. These criteria can also be simply adapted to multi-player games. As we will see, the strategies are sufficient to ensure very high probabilities of winning.

2 The game of Texas Hold'em Poker

We will attempt in this study to consider a type of poker game as close as possible to that of the *Texas Hold'em no limit* as it is officially described in [6] and the principles of which are laid out in this section.

2.1 The game rounds

Texas Hold'em Poker is played with a standard 52-card deck and each game is punctuated by an alternation of card dealing and rounds of betting. There are 4 of these phases and card dealing takes place in the following order.

Preflop: Each player is dealt 2 pocket cards face down.

Flop: Three community cards (the flop) are now dealt face up.

Turn : A fourth community card (the turn) is revealed

River : A fifth community card (the river) is revealed

At the end of each phase of card dealing, a betting round starts. Players place their bets one after the other and if a player wishes to stay in the game, he or she must at least match the biggest stake. This round of betting stops when all the players who are still in the game have bet the same number of chips. All the bets make up what is called **the pot**. Finally two scenarios are possible: either there is only one player left in the game and he or she wins the pot or there are several players left and the game moves to the next stage.

If, after the last round of betting following the river, several players are still in the game, they are rewarded according to the value of their hand. Each player can then make use of his or her 2 pocket cards and the 5 community cards to make the best 5-card hand possible out of the 7 available. The player with the best hand then wins the pot and, in the event of two or more hands being worth the same, the players concerned split the pot. The various possible poker hands are described in the following section.

A game of poker is made up of a succession of rounds of this type, rounds where players take it in turns to bet first. At the beginning players have the same amount of chips available to them and they fold when they have no chips left. In order to encourage players to bet, compulsory stakes are added at the first round of betting (**blind** or ante) for certain players on the table.

2.2 Hand ranking

At the end of a poker game, each player still in the game must reveal his or her cards and the strength of his or her cards is determined by the best 5-card hand he or she can assemble out of the 7 available to him or her. In a 52-card deck, there are 2,598,960 5-card unordered hands possible and all these combinations of cards are separated in ten categories according to their probability of appearing:

Royal Flush: Ace, King, Queen, Jack and Ten of the same suit.

Straight Flush: Any straight with all 5 cards of the same suit.

Four of a kind: 4 cards of the same rank.

Full house: 3 cards of the same rank together with any 2 cards of the same rank.

Flush: 5 cards of the same suit which are not consecutive.

Straight: 5 consecutive cards of different suits.

Three of a kind: 3 cards of the same rank.

Two-pair: 2 cards of the same rank together with another two cards of the same rank.

One-Pair: 2 cards of the same rank.

High card: Any hand that does not make up any of the above-mentioned hands.

The rarer a 5-card hand is the more it is worth. Within each category, hands are ranked according to how high the cards are. All 5-card hands can therefore be ranked amongst each other, with sometimes the possibility of a draw. Let us now consider combinations of 7 cards where the best 5-card hand out of the 7 possible is kept. There are then 133,784,560 possible combinations of 7 unordered cards. The following table shows [1] and [9], for each hand category, the number of 5 and 7-card possible combinations and their probability to occur.

	5-card combination		7-card combination	
	Number	Probability in %	Number	Probability in %
Royal Flush	4	$1.5\ 10^{-4}$	4324	$3.2\ 10^{-3}$
Straight Flush	36	$1.4\ 10^{-3}$	37 260	$2.8\ 10^{-2}$
Four of a kind	624	$2.4\ 10^{-2}$	224 848	0.17
Full house	3 744	0.15	3 473 184	2.60
Flush	5 108	0.20	4 047 644	3.03
Straight	10 200	0.39	6 180 020	4.62
Three of a kind	54 912	2.11	6 461 620	4.83
Two-pair	123 552	4.75	31 433 400	23.5
One-Pair	1 098 240	42.3	58 627 800	43.8
High card	1 302 540	50.11	23 294 460	17.4

3 Analysis of N. Alon's article

3.1 Structure of the analysis

In his analysis [1], N. Alon considers a variant of poker game suggested by Von Neumann and Morgenstern [5]. He considers a game with two players, Alice and Bob which works as follows:

1. Each player is 'dealt' a random number drawn in even interval [0, 1]. This number represents the value of their cards and is represented with an x_A for Alice and x_B for Bob.
2. Both players then decide simultaneously to bet 1 chip or to fold.
3. If one of the two players has folded, the game stops and there is no exchange of money. If both players have bet, the player whose cards have the highest value wins the amount wagered by the other, i.e. 1 chip.

This simple game variant captures the essence of the last round of a poker game well, when all the community cards are revealed. Indeed, there are then $C^2_{47} = 1081$ possible combinations of two cards for each player. Disgarding the draws and superpositions between the various combinations, these combinations can be ordered. Each player is therefore dealt cards with a value ranging from 1 to 1081. This range, divided by 1081, resembles the values of the card combinations x_A and x_B considered in the variant. At the end of the game, if both players have bet, the player with the highest range wins the pot. The benefit of considering such a simplified game is that it allows us to perform analytical calculations in an explicit manner and to calculate the players expected winnings.

N. Alon mainly considers a two-player game where Bob plays in a random manner and Alice plays in a more strategic way by adapting to Bob's game. Bob's random behaviour can be represented with the following pattern: he bets 1 chip with a probability of $p := 1/2$ and folds therefore with a probability of $1 - p = 1/2$. Alice knows Bob's strategy and seeks to adapt her strategy to his behaviour.

3.2 The Results

N. Alon then demonstrates that the optimal strategy for Alice consists in betting if and only if the value of her cards is above 1/2. He then shows that her winning odds are 1/8 at each game and that the associated variance is 15/64. In other words, Alice's average winning odds at each game are 1/8 and the manner in which they fluctuate is characterised by a variance of 15/64.

Thanks to the Central Limit Theorem, he can then estimate the probability of Alice losing following a sufficiently high number n of consecutive games. The results are very convincing; after for example 350 games, he observes that the number of times Alice looses is less than 1 in a million. He thus naturally comes to the conclusion that for a sufficiently high number of games, strategy is a determining factor in the outcome of a game.

He also broadens these results by adding compulsory bets (blinds) in each game and he briefly analyses the instance of a game with more than 2 players.

3.3 The main limitations

The results presented in N. Alon's article are absolutely valid and pertinent but it is nevertheless regrettable that the analysis should be restricted to a simplified version of poker game which only takes the last round of the game into account. It is true that the analytical calculations proposed would be difficult to apply to the complete version of the game. We will therefore broaden N. Alon's findings to a real poker game model by replacing the analytical calculations with digital simulations.

In the same way, the fact that Alice knows Bob's strategy, in other words the probability p with which he bets one chip, is a problem. Indeed, it *is* rather surprising that she would be able to identify her opponent strategy that easily. Here, we will aim to construct reasonable strategies which seem adapted to Bob's various random ways of playing.

4 The model considered

We will therefore use a similar approach to that of N. Alon, but with a game very close to the real rules of *Texas Hold'em Poker*. In the first instance, in order to better understand the required strategies, we will limit ourselves to games with 2 players.

4.1 The game

We will consider that two players, Alice and Bob are sat around a table of poker.

4.1.1 Stages of the game

The game progresses in 4 rounds:

Preflop: Alice and Bob are dealt 2 cards each. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, the game continues.

Flop: the first three community cards are dealt face up. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, the game continues.

Turn: the fourth community card is revealed. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, the game continues.

River : the fifth community card is revealed. They simultaneously decide to bet 0 or 1 chip. If both players bet the same amount, they compare hands. The player with the best hand wins the pot or in the event of a draw, the two players share it.

Note that the main differences between a real *Texas Hold'em* Poker game and this version are as follows:
- There is no compulsory bet or blind
- The amounts wagered at each round are fixed and equal to 1
- At each round the players decide simultaneously whether they want to bet
- It's a 2-player game

We will include in section 4.3 some variants of the game described here and they will take the following differences into account: blind, variable bets, several players.

4.1.2 The players

Just like in N. Alon's article, we consider that 2 people play against each other:
- Bob who has a random strategy with a probability p of betting and (1 -p) to fold at each round of the game.
- Alice who aims to adapt her strategy to Bob's. She suspects that Bob's strategy is random but she does not know the probability p which governs his decisions. She makes her decisions by estimating what his pocket cards might be.

4.2 Alice's optimal strategy

Because Alice does not know the probability p characterising Bob's behaviour, she will devise a strategy not dependent on p. She observes however that at the last round where the pot was not nil, the situation most favourable to Bob is if he never folds, i.e. p=1. To define her strategy, she therefore considers the case where Bob never folds. It is this strategy which she will use later even when p is different from 1.

4.2.1 The optimal strategy during the River

Alice's optimal strategy during the river is easy to determine. Let us assume that the pot is worth P, and that Alice has cards which mean that the probability that she will win is X. Knowing that Bob does not fold and discarding the possibilities of a draw, if Alice bets, her chance of winning is $X(P + 2) - 1$. Her chance of winning being nil if she folds, we can deduce that

$$\text{Alice must bet at the river if and only if} \qquad X \geq \frac{1}{P+2}$$

The interpretation of this boundary is clear and can be read in the following manner: as she wagers one chip in the hope of winning $P + 2$, it is in her interest to play if and only if her probability of winning is greater than $1/(P + 2)$. Note that the more there is in the pot, the least important it is for Alice to have good cards in order to bet.

To estimate her probability of winning X, all Alice has to do is count the number of hands she could beat amongst the $C^2_{45} = 990$ other possible hands. This calculation, easily performed by a computer is of course impossible for a human brain. However for experienced players, it is not difficult to estimate X relatively precisely. In order to adapt to the reality of a player who estimates his or her probability X of winning with possibly one error, we will present in section 5.2.4 the results of digital games where we have artificially added a random measuring error on the estimation of X.

4.2.2 The optimal strategy during the Turn

Let us assume that Alice uses in the last round the strategy previously described. Let us then work out what her strategy should be at the previous round. Note that P is the amount in the pot and X is the random variable equal to Alice's winning odds at the last round of the game. The variable X is random in the sense that it is not yet known; it can indeed have 46 different values depending on the last community card. If Alice decides to bet in this round, her odds of winning are:

$$\mathrm{E}\left[(X(P + 4) - 1)1_{X\geq 1/(P+4)} - 1\right].$$

Accordingly, as she would win nothing by folding, it is in Alice's interest to bet if and only if

$$\mathbb{E}\,[X1_{X\geq 1/(P+4)}] \geq \frac{1 + \mathbb{P}[X \geq 1/(P+4)]}{P+4}$$

This time, Alice's strategy which when analysed appears more complex is in fact also very intuitive. The principle is the following: There is no point in Alice betting at this round if she does not bet at the following one. Therefore she hopes to win $P + 4$ by betting 1 at this round and 1 at the following. She is testing whether her odds of winning by betting at the last round are greater than the total she has wagered divided by her winnings. Calling R the event where Alice bets during the river,

Alice must bet during the Turn if and only if $\mathbb{E}\,[X1_R] \geq \dfrac{1 + \mathbb{P}[R]}{P+4}$

4.2.3 Optimal strategy during the Flop

The same type of rationale can be applied to the flop if a strategy has been defined thereafter. Calling T, the event where Alice bets during the turn and $T \cap R$, the event where Alice bets both during the turn and during the river, Alice's odds of winning pot P are the following:

$$\mathbb{E}\,[(X(P+6) - 1)1_{T\cap R} - 1_T - 1]\ .$$

One can see that Alice will have to bet if and only if her winning probability is greater than the ratio between the potential amount she has wagered and the total pot.

Alice must bet during the Flop if and only if $\mathbb{E}\,[X1_{T\cap R}] \geq \dfrac{1 + \mathbb{P}[T\cap R] + \mathbb{P}[R]}{P+6}$

4.2.4 Optimal strategy during the Preflop

Obviously, the same rationale always apply when none of the community cards are yet available. Calling F the event where: "Alice bets during the Flop" we know that

Alice must bet during the Preflop if and only if $\mathbb{E}\,[X1_{F\cap T\cap R}] \geq \dfrac{1 + \mathbb{P}[F\cap T\cap R] + \mathbb{P}[T\cap R] + \mathbb{P}[R]}{8}$

4.2.5 Optimal strategy when the stakes vary

So far we have assumed that the amount wagered at each round equalled 1 chip. Of course in reality, stakes vary. Taking into account the variability of these stakes renders the previous analysis much more complex. Indeed in this instance the criterion used whereby the odds of winning are maximised does not seem very appropriate. However the results previously obtained can be easily extended to the scenario where the stakes at each round are different but determine ,or vary according to, the pot. Empirically, in a poker game, the closer the river is the more the stakes go up.

Let us call m_P, m_F, m_T and m_R the stakes for each player respectively during the preflop, during the flop, during the turn and during the river. We deduce that the optimal criteria which Alice must consider at each round of the game for a pot of P are the following:

River: $X \geq \dfrac{m_R}{}$

$$\frac{\qquad}{P + 2\, m_R}$$

Turn: $$\mathbf{E}[X1_R] \geq \frac{m_T + m_R \mathbf{P}[R]}{P + 2 * (m_T + m_R)}$$

Flop: $$\mathbf{E}\,[X1_{T \cap R}] \geq \frac{m_F + m_T \mathbf{P}[T \cap R] + m_R \mathbf{P}[R]}{P + 2 * (m_F + m_T + m_R)}$$

Preflop: $$\mathbf{E}\,[X1_{F \cap T \cap R}] \geq \frac{m_P + m_F \mathbf{P}[F \cap T \cap R] + m_T \mathbf{P}[T \cap R] + m_R \mathbf{P}[R]}{P + 2 * (m_P + m_F + m_T + m_R)}$$

4.3 The strategies put into action

In order to make this analysis more realistic and more reader-friendly, we will simplify the criteria taken into account in Alice's decision making process. We will therefore not use the optimal strategy which Alice has available to her. Here we will consider only strategies that are merely based on Alice's hand odds of winning versus another hand. In other words, in each round, Alice knows $\mathbf{E}\,[X]$ where X represents her probability of winning at the last round of the game. Of course, $\mathbf{E}\,[X]$ is recalculated at each round of the game by taking into account the new community cards. For an experienced player, it is easy to know the potential of one's hand and to see how it evolves as the flop, the turn and the river unfold.

4.3.1 The reference strategy

We will consider strategies where, at each round of the game,**Alice bets if and only if E [X] > x(P)** where P is the value of the pot and x a decision function. We still have to specify the functions x_P, x_F, x_T et x_R corresponding to each round of the game: preflop, flop, turn and river. To simplify the presentation, we will assume that the stakes at each round are equal to one chip.

First of all, during the river, X is no longer random as all the cards are known and the criteria of optimal strategy can be applied:

$$X \geq \frac{1}{P+2} \qquad \text{namely} \qquad x_R(P) := \frac{1}{P+2}$$

Let us now consider the other rounds of the game. The previous optimal criteria can not be applied but they give a good idea of the threshold function x_P, x_F et x_T that is reasonable to use. Let us assume that at each round of the game, Alice decides to bet thinking that she will not fold at any of the following rounds. Then her decision criteria during the river, the flop and the preflop become respectively

$$\mathbf{E}[X] \geq \frac{2}{P+4}, \qquad \mathbf{E}[X] \geq \frac{3}{P+6} \quad \text{and} \quad \mathbf{E}[X] \geq \frac{4}{P+8}$$

The criteria therefore have the desired make up and will be the ones we adopt. Note also that the pot is inevitably empty when the players are at the preflop so $x_P := x_P(0) = 1/2$. Given that the pot has a value of P, we will in fact use Alice's following decision criteria:

Preflop: $\mathbf{E}[X] \geq x_P := \frac{1\ ;}{\qquad}$

$$\overline{\qquad\qquad 2}$$

Flop: $E[X] \geq x_F(P) := \dfrac{3}{P+6}$;

Turn: $E[X] \geq x_T(P) := \dfrac{2}{P+4}$;

River: $X \geq x_R(P) := \dfrac{1}{P+6}$.

We will now deal with the distinct variants of the game allowing us to better take into account the *Texas Hold'em Poker* game specifics. We provide strategies when the stakes are varied, when a blind is added or when there are more than 2 players.

4.3.2 Strategy with increasing stakes

In a game of Poker, stakes empirically appear to increase with each round of the game. We have already presented the optimal strategy with varied stakes in section 4.2.5. Let us call m_P, m_F, m_T and m_R the stakes at preflop, flop, turn and at the river respectively. These stakes are known in advance as being dependent on the pot. By adapting the rationale described in the previous section, for a pot that is worth a given value of P, the following criteria are easily obtained:

Preflop: $E[X] \geq x_P := \dfrac{1}{2}$;

Flop: $E[X] \geq x_F(P) := \dfrac{m_F + m_T + m_R}{P + 2 * (m_F + m_T + m_R)}$;

Turn: $E[X] \geq x_T(P) := \dfrac{m_T + m_R}{P + 2 * (m_T + m_R)}$;

River: $X \geq x_R(P) := \dfrac{m_R}{P + 2 * m_R}$.

Alice's strategy is rather cautious. Intending to play until the last round, Alice chooses a relatively high level of cards to bet.

4.3.3 Strategy with a blind

In the game considered so far, players decide simultaneously whether they want to bet or not and they do not have a forced bet (blind). Let us consider a game where each player is forced to wager 1 chip at preflop every other game. This way, folding has an additional cost and the 2 player's roles are asymmetrical. So we isolate 2 cases depending on whether it is Bob or Alice who pays the blind.

1. Bob pays the blind: then Alice's strategy which is based on the fact that Bob will in any case bet, will not change.
2. Alice pays the blind: in this case Alice is forced to bet at the preflop. She then applies her strategy simply from the flop.

To conclude, Alice's strategy remains unchanged, apart from the fact that every other time, she has no choice but to bet at the preflop.

4.3.4 Strategy for a game with n players

We consider here a game where Alice has n opponents who have the same strategy as Bob. These n players bet with a probability of p and fold with a probability of 1-p. Because she does not know p, Alice seeks a strategy based on the assumption that the other players always bet. All the same we assume that she will adapt her strategy to the number of players still in the game at each round.

Let us say that we are at the river with a pot of a value of P and a number n of players still in the game. Alice knows X, the probability of her hand beating another hand given the five community cards. Accordingly, discarding cards superposition, her probability of beating all n other players equates to X^n. As she wagers 1 chip in the hope of winning $P + n + 1$, her criterion of choice is obtained by the operation: $X^n \geq 1/(P + n + 1)$.

The same type of rationale can be applied to the various rounds of the game. The probability X of winning at the last round can simply be replaced by X^n and in the event of winning, the winnings can be adapted to the number of players n still in the game. Calling the different stakes at each round m_P, m_F, m_T and $m_{R,}$ one obtains for a pot of a value of P and n players still in the game the following criteria:

Preflop: $\mathrm{E}[X^n] \geq x_P := \dfrac{1}{2}$;

Flop: $\mathrm{E}[X^n] \geq x_F(P) := \dfrac{m_F + m_T + m_R}{P + (\mathrm{n} + 1) * (m_F + m_T + m_R)}$;

Turn: $\mathrm{E}[X^n] \geq x_T(P) := \dfrac{m_T + m_R}{P + (\mathrm{n} + 1) * (m_T + m_R)}$;

River: $X^n \geq x_R(P) := \dfrac{m_R}{P + (\mathrm{n} + 1) * m_R}$.

5 Digital tests

In his analysis, N. Alon calculates in a theoretical manner Alice's winning odds and the variance Y at each game where she uses her strategy. With these odds and variance, he deduces from the Central Limit Theorem the odds of Alice loosing after any random sufficiently high number *n* of games. In our more general game model, we cannot exactly calculate the odds and variance of this random variable *Y*. We have therefore chosen to perform digital simulations of virtual games in order to estimate them. These estimation techniques called the 'Monte Carlo' methods are represented in section 5.1.2

In order to implement Alice's strategy in a computer environment, we had to, at each round of the game, calculate her probability of winning $\mathbb{E}[X]$. In order to calculate it at the river, at the turn and at the flop, we have simulated all the possible combinations of cards being revealed so that this probability can be calculated precisely. On the other hand, in order to calculate it during the preflop, when Alice only knows her two pocket cards, we have used tables of already calculated probabilities. These tables depend of course on the number of players and have been taken from [2] and [8]. In order to make Alice's strategy more realistic, we also present digital results where errors on the calculation of $\mathbb{E}[X]$ have been artificially introduced.

In this section, we first tackle the mathematical theorical justifications underlying our approach, then we present the digital results obtained.

5.1 Theoretical justification

5.1.1 The Central Limit Theorem

We lay out the manner in which, from Alice's winning odds and variance over a game, we can work out her odds of losing after a given number n of games. The result expressed had already been noticed and used by N. Alon [1].

We start by presenting (a version of) the central limit theorem.

Theorem 5.1 *Be M a real positive number and Y1, Y2, ... a suit of random variables independent of such laws in such a way that each Y_i satisfies $|Y_i| \leq M$. Calling μ et σ^2 the odds and the variance of each X_i, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i = \mu\,, \qquad \mathbb{P} - p.s.$$

In addition, we have

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}} \leq z\right] = \Phi(z)\,, \tag{5.1}$$

where Φ is the distribution function of normal law defined by :

$$\Phi(z) \quad := \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}dt\,.$$

Let us assume that Y_i represents Alice winning at the ith game. Then, $\sum_{i=1}^{n} Y_i$ is the sum of Alice's winnings over the first n games. Accordingly, Alice will be losing after *n* games if and only if $\sum_{i=1}^{n} Y_i \leq 0$. As the random variables Y_i are independent and of the same law; we can apply the previous theorem and deduce the following result:

Proposition 5.1 B*e μ and σ^2 Alice's winning odds and variance at each game. The odds of Alice losing after a sufficiently high number n of games is in the order of* $\Phi\left(-\mu\sqrt{n}/\sigma\right)$.

In his analysis, N. Alon manages to calculate Alice's winning odds and variance at each game perfectly. In our more general game model, we cannot do this so we will obtain a digital approximation thanks to the Monte Carlo method.

5.1.2 The Monte Carlo methods

Given a specific strategy for Alice, we are seeking to estimate the odds μ and the variance σ^2 of her winning Y at each game. The Monte Carlo methods are based on the Central Limit Theorem stated above. Let us consider a pool of n games where Alice's winnings Y_i are available to us. Then, according to (5.1) in the theorm 5.1, we can estimate μ with

$$\hat{\mu}_N := \frac{1}{N}\sum_{i=1}^{N} Y_i \, .$$

Note that a standard estimator of Y 's variance is given by

$$\hat{\sigma}_N^2 := \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \frac{1}{N}\sum_{i=1}^{N} Y_i\right)^2 \, .$$

The idea is to use $\hat{\mu}_N$ and $\hat{\sigma}_N^2$ instead of μ and σ^2. We can demonstrate that the results stated in theorem 5.1 stay true when the variance σ^2 is replaced by its estimation $\hat{\sigma}_N^2$, see [4] for example. From this we conclude the following result:

Proposition 5.2 *Let us consider a pool of N poker games where, for each $i \leq N$, Y_i represents Alice's winnings at the i^{th} game. Then, the possibility of Alice losing after a sufficiently high number n of games is in the order of* $\Phi\left(-\hat{\mu}_N\sqrt{n}/\hat{\sigma}_N\right)$, *with*

$$\hat{\mu}_N := \frac{1}{N}\sum_{i=1}^{N} Y_i \, , \quad et \quad \hat{\sigma}_N^2 := \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i - \frac{1}{N}\sum_{i=1}^{N} Y_i\right)^2 \, .$$

5.2 Digital results

Here we present the digital results obtained in simulations of a pool of 30,000 poker games played between Alice and one or several random players. The strategies employed are those detailed in section 4.3. They were elaborated in the context of the opponent betting at all the rounds but we will test them in the context of the opponent betting randomly with a probability of p.

Alice's winning odds and variance are unknown and are therefore estimated with the Monte Carlo methods previously described. There is therefore a measuring error on the magnitude of these numbers which is absolutely controllable. The values provided below are not absolutely exact but the important thing is that their order of magnitude are completely valid.

In each variant of the game, the conclusion remains the same: for a sufficiently high number of games, strategy is a deciding factor in the outcome of a game.

5.2.1 The reference game

Let us consider first of all the reference game for which Alice's strategy has been presented in section 4.3.1, with a stake of 1 chip at each round. The following table provides Alice's estimated winning odds and variance at each game for various values of p. With proposition 5.2, we also calculate the odds of Alice losing after 50, 100 and 500 games.

P		1	0.9	0.8	0.7	0.6	0.5
Odds		0.32	0.35	0.36	0.36	0.33	0.30
Variance		6.7	5.2	3.9	2.8	2.0	1.4
% of chance for Alice to not be leading the game after...	50 games	18.7	15.7	12.2	8.7	5.3	2.5
	100 games	10.5	7.7	5.0	2.7	2.10^{-2}	2.10^{-3}
	500 games	0.25	0.07	0.02	9.10^{-4}	2.10^{-5}	3.10^{-8}

It's very clear that Alice's strategy gives her a considerable advantage. Figure 1, representing, for various values of p, the odds of Alice having lost money after n games, is also very telling. In the most unfavourable case where $p = 1$, the number of times when Alice is in a losing position is 1 in 10 after 100 games, less than 3 in a thousand after 500 games, and less than 4 in 100,000 after 1,000 games.

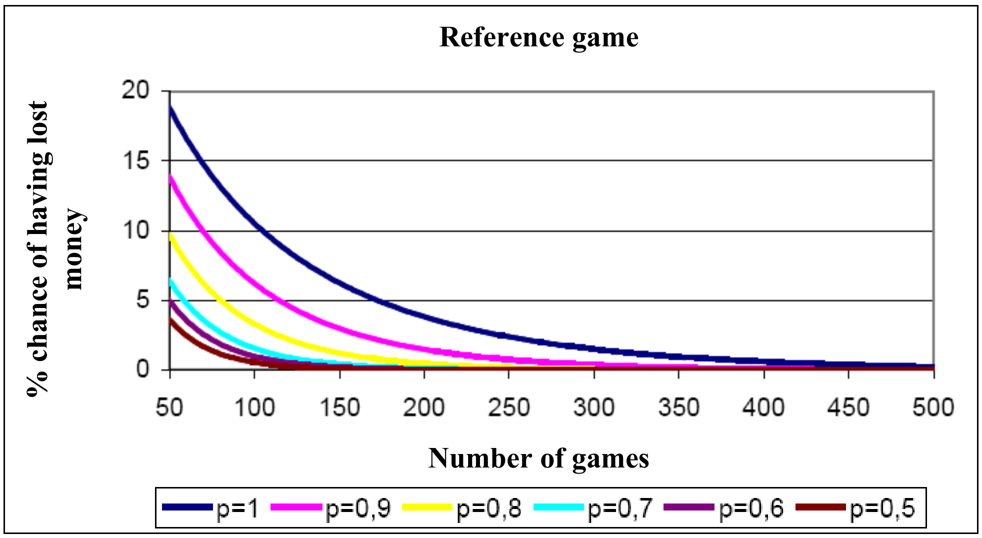

FIG. 1 - % chance for Alice to be in deficit vs the number of games

In a game of poker, each player has available to him or her the same initial number of chips and gets ‘knocked-out’ when he or she has no chips left. So in order to get knocked-out, a significant number of games have to be lost without winning too many of them. In this reference game there are exchanges of at most 4 chips at each round . After grading Alice's winnings distribution for a game, we have also estimated the odds of Alice losing given a specific initial number of chips. In the worst case scenario where $p = 1$, the results shown in the following table are very convincing. The higher the initial number of chips is, the more games Alice has to loose in order to be knocked-out. Therefore Alice gets knocked-out less often. We can observe for example that the number of times Alice looses is only 6 out of 1,000 with a fairly reasonable initial number of 50 chips.

Initial number of chips	10	25	50	100
Odds of Alice being knocked-out	24%	7%	0.6%	0.005%

5.2.2 Increase of stakes

Let us now consider the extended scenario where the stakes increase at each round of the game. Alice's strategy in this case is presented in 4.3.2. We are still making the assumption that Bob bets with probability p and we present the digital results for cases where the stake is 1 at preflop, 2 at the flop, 4 at the turn and 8 at the river. The digital results are presented in the following table and in Figure 2.

P		1	0.9	0.8	0.7	0.6	0.5
Odds Variance		1.5 61	1.3 50	1.1 41	1 33	0.8 25	0.7 19
% of chance for Alice to not be leading the game after...	50 games 100 games 500 games	8.9 2.9 10^{-3}	9.9 3.4 2.10^{-3}	10.8 4.0 4.10^{-3}	11.2 4.2 6.10^{-3}	11.8 4.7 9.10^{-3}	12.0 4.9 10^{-2}

In this version of the game, Alice has again a considerable advantage over her opponent. However, her winning variance is high because as many as 30 chips can be wagered in this game. Note that the winning odds decrease with *p*. Indeed Alice has a rather cautious strategy and the more often Bob folds, the less lucrative the games are for her, insomuch as the important stakes are at the end of a game. Even in the most unfavourable case analysed here, Alice's odds of losing are less than in the previous reference game. In the case where $p = 1$, the number of times when Alice is in a losing position is 3 in 100 after 100 games, 1 in 100,000 after 500 games, and less than 1 in a million after 650 games.

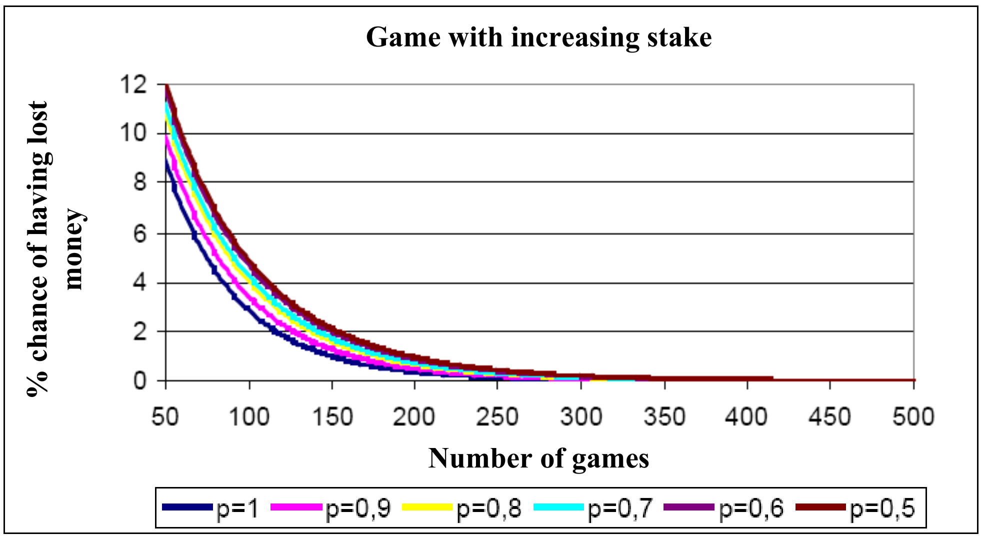

FIG. 2 - % chance for Alice to be in deficit vs number of games

5.2.3 Blind added

An important component in the rules of *Texas Hold'em Poker* is the use of forced stakes, a.k.a. 'the blind'. They force players to wager and put emphasis on their position around the table. Considering games with a forced stake alternating between the two players of 1 chip at the preflop, we have carried out a digital test of the strategy presented in section 4.3.3. In the following table, we show Alice's estimated winning odds and variance. As previously, the results are complemented by the odds of Alice not leading after n games, see Figure 3.

P		1	0.9	0.8	0.7	0.6	0.5
Odds Variance		0.28 8.4	0.32 6.7	0.36 5.2	0.36 4.0	0.39 2.9	0.38 2.1
% of chance for Alice to not be leading the game after...	50 games 100 games 500 games	24.9 16.9 1.6	18.7 10.4 0.24	13.2 5.7 2.10^{-2}	8.5 2.6 7.10^{-4}	5.4 1.1 2.10^{-5}	3.2 0.4 2.10^{-7}

Unsurprisingly, Alice's performance is not as good in this game model as she sometimes has to wait for the second round of the game before she can fold despite the fact that she may have a bad hand. Like in the previous two games, strategy is nevertheless still the overiding factor in the outcome of a game. Indeed, in the most unfavourable case where $p = 1$, the number of times that Alice is in a losing position is 17 in 100 after 100 games, less than 2 in 100 after 500 games and less than 1 in a thousand after 1,000 games.

In order to compare this game with the reference game, we have also graded Alice's winnings distribution for a game and estimated the odds of Alice losing given an initial number of chips. In the worst case scenario where p = 1, the results are shown in the following table. We can observe for example that the number of times Alice looses in this instance is 3 in 100 for an initial number of 50 chips.

Initial number of chips	10	25	50	100
Odds of Alice being knocked-out	32%	15%	3%	0.01%

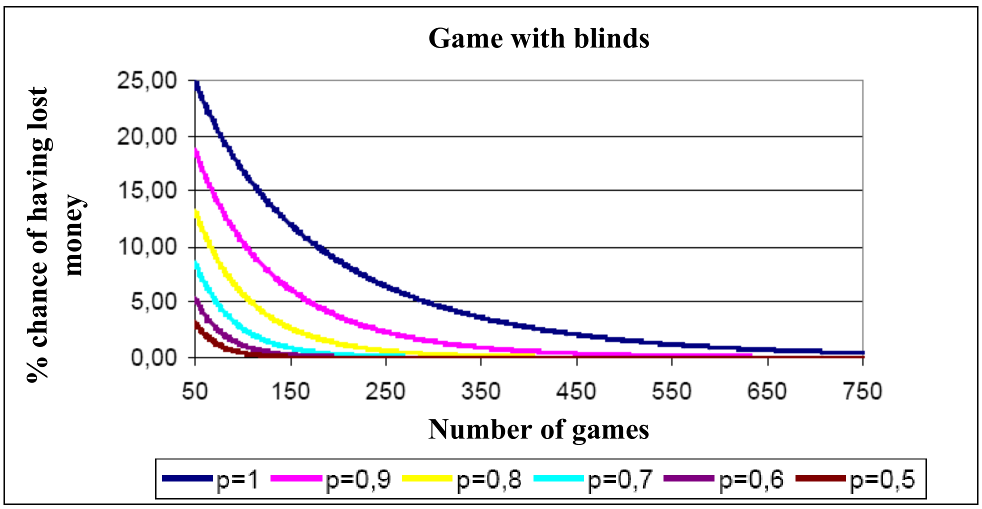

FIG. 3 - % chance for Alice to be in deficit vs number of games

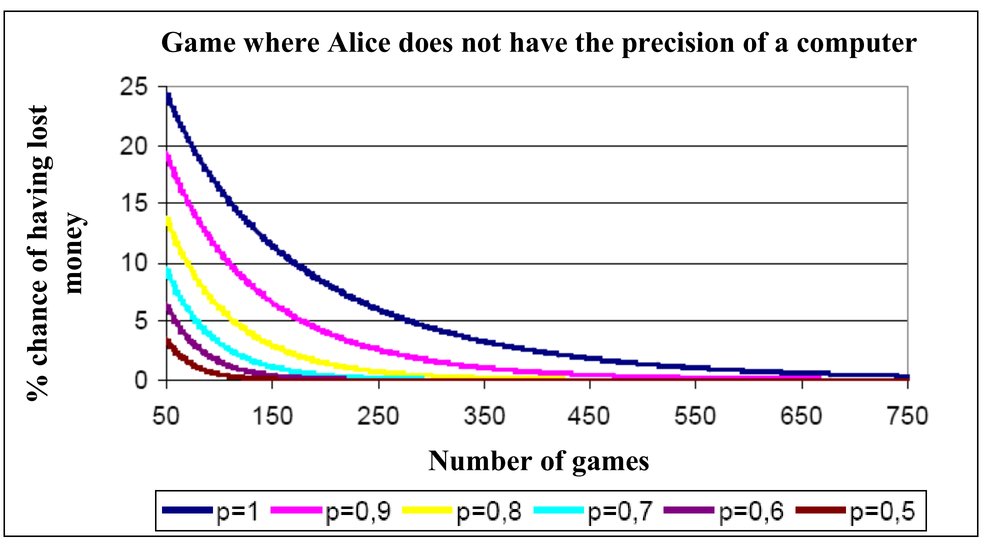

FIG. 4 - % chance for Alice to be in deficit vs number of games

5.2.4 Alice does not have the precision of a computer

In all the strategies employed until now, we always assume that Alice knows how to perfectly evaluate her hand. An experienced player certainly always has a precise idea of his or her 'hand in the pocket' potential, however it does not seem reasonable to expect a player to be able to evaluate his or her hand in such a precise manner. In order to make this analysis more realistic, we have artificially added a random error on the evaluation that Alice makes of her hand. We have mathematically added the odds of winning E [X] calculated by Alice with an independent centered normal law with a variance of 10^{-2}. In other words Alice now estimates her odds of winning and her approximation has a precision of 0.2, 95 times out of 100. For example, if Alice's odds of winning are 0.6, she will estimate them between 0.4 and 0.8 and will adapt her strategy to her estimation. The results obtained in this way are laid out in the following table and in Figure 4.

P		1	0.9	0.8	0.7	0.6	0.5
Odds Variance		0.23 5.5	0.27 4.7	0.30 3.9	0.33 3.2	0.35 2.6	0.36 2.1
% of chance for Alice to not be leading the game after...	50 games 100 games 500 games	24.4 16.3 1.4	19.3 11.0 0.3	13.8 6.2 2.10^{-2}	9.5 3.2 3.10^{-3}	6.5 1.7 9.10^{-5}	3.5 0.5 5.10^{-7}

We arrive to the same conclusions as previously: even if Alice wins less often because she badly estimates her hand potential, her strategy remains dominant over that of her opponent. This way, in the most unfavourable case where $p = 1$, the number of times when Alice is in a losing position is 16 in 100 after 100 games, less than 2 in 100 after 500 games, and less than 1 in 1,000 after 1,000 games.

In this game model, we have also estimated Alice's odds of losing given an initial number of chips. In the worst case scenario where $p = 1$, the results are shown in the following table. In this instance, the number of times that Alice looses is 1 in 100 for an initial number of 50 chips.

Initial number of chips	10	25	50	100
Odds of Alice being knocked-out	28%	10%	1.3%	0.02%

5.2.5 A 4-player game

We now consider a 4-player game, where Alice plays against 3 players who have a random strategy characterised by p. Alice uses the strategy laid out in section 4.3.4. We have performed a digital estimation of Alice's winning odds and variance. In a game of 4 players, proposition 5.2 allows the calculation of Alice's odds of losing money after n games. The results obtained for various values of p are laid out in the following table and in Figure 5.

P		1	0.9	0.8	0.7	0.6	0.5
Odds Variance		0.60 20	0.88 17	0.89 12	0.86 8	0.82 6	0.69 4
% of chance for Alice to not be leading the game after...	50 games 100 games 500 games	17.2 9.1 0.1	6.4 1.5 7.10^{-5}	3.5 0.5 5.10^{-7}	1.8 0.2 2.10^{-9}	0.8 3.10^{-2} 7.10^{-13}	0.5 10^{-2} 2.10^{-14}

Once again, the conclusion is similar: for a sufficiently high number of games played, the players' results are very clearly correlated with their respective strategies. Thus, in the most unfavourable case where p = 1, the number of times that Alice loses money is around 9 in 100 after 100 games, 1 in a thousand after 500 games, and 1 in 100,000 after 1,000 games.

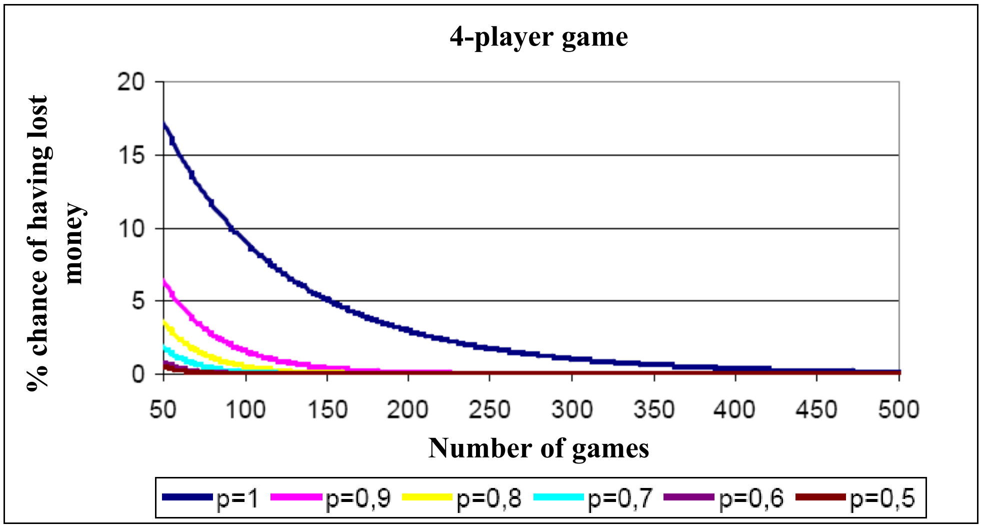

FIG. 5 - % chance for Alice to be in deficit Vs Number of games

Conclusions

Here we have analysed the influence of chance on the outcome of poker games between several players, one player having a dominant strategy over the others. In order to determine this dominant strategy, we have assumed that the other players were playing randomly and we have carried out a theoretical analysis over the expected winnings of a strategical player. Stemming from this analysis, we opted for a strategy which was not optimal but which was easy to understand.

In order to quantify the performances of a strategic player versus his opponents, we have performed a computer simulation of a pool of virtual poker games. This has enabled us to evaluate the winning odds and variance of a strategic player and to work out his or her chances of winning. We have considered game cases with 2 or 4 players, with or without blind, with constant or variable stakes. We also studied the case where the strategic player estimates his or her hand potential with little precision.

In all the game variants, the conclusion remains the same: for a sufficiently high number of consecutive games of *Texas Hold'em Poker*, the quality of the strategy employed has an overiding influence over the outcome of the game. Our conclusions are therefore similar to N. Alon's [1] but we have also dealt with broader game models. Furthermore these conclusions are completely consistent with the empirical observation that it is usually the same professional players who reach the final phases of Poker tournaments.

References

[1] Alon N. (2007). Poker, Chance and Skill. Preprint.

[2] Montmirel F. (2007). Poker Cadillac, *Fantaisium*.

[3] Cutler W. (1975). An optimal strategy for Pot-limit Poker, *The American Mathematical Monthly*.

[4] Fishman G.S. (1995). Monte Carlo concepts, algorithms and applications. *Springer series in Operation Research*.

[5] Von Neumann J. & O.Morgenstern (1944). Theory of Games and Economic Behaviour, *Princeton University Press*.

[6] Arrêté du 14 mai 2007 relatif à la règlementation des jeux dans les casinos (Decree of the 14th of May 2007 regarding the regulation of games in Casinos), (2007) Official *Journal of the French Republic*.

[7] Kelly J. M., Dhar Z. & T. Verbiest (2007). Poker and the Law : Is it a Game of Skill or Chance and Legally Does it Matter ? *Gaming Law Review* 11(3), 190-202. doi :10.1089/glr.2007.11309.

[8] Poker, rules and strategies. "http ://www.poker-regle/strategie.com/" rubrique "probabilites".

[9] Poker on Wikipedia. "http ://fr.wikipedia.org/wiki/Poker".